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LETTER TO THE EDITOR

Some exact solutions for the potential $-r^{-1}+2\lambda r+2\lambda^2r^2$

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Abstract. We show how to obtain an infinite number of exact solutions for the excited states of an s-wave hydrogen atom with the polynomial perturbation $2\lambda r+2\lambda^2r^2$ for certain specific values of λ. The energy eigenvalues in such cases satisfy $E_n = -\frac{1}{2}+(2n+3)|\lambda|$ with the corresponding wavefunctions being given by products of $\exp(-|\lambda|r/\lambda-|\lambda|r^2)$ and polynomials of nth degree in r.

The s-wave Hamiltonian for a hydrogen atom with a polynomial perturbation given by

$$H=-\frac{1}{2}\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}-\frac{1}{r}+2\lambda r+2\lambda^2r^2 \tag{1}$$

was first studied by Killingbeck (1978) who pointed out that the system possesses an exact solution for the ground state for $\lambda>0$ and that the Rayleigh–Schrödinger perturbation series for the ground-state energy in powers of λ was not valid for $\lambda<0$. In a recent paper (Saxena and Varma 1982), we have explained why such a breakdown is expected to occur and have constructed a perturbation expansion for the ground-state energy in powers of $|\lambda|^{-1/2}$ valid for $\lambda<0$ which gives good agreement with eigenvalues computed numerically.

In this letter we wish to point out the existence of an infinite number of exact solutions for the excited states of the Hamiltonian (1) for certain specific values of the coupling λ. We do this separately for positive and negative λ using a method which has recently been reported by one of us (Varma 1981).

Case I. $\lambda>0$

In the Schrödinger equation corresponding to the Hamiltonian (1) we substitute the following ansatz for the wavefunction

$$\begin{aligned}\psi(r)&=\phi(r)\exp(-r-\lambda r^2)\\ \phi(r)&=\sum_{m=0}^{\infty}a_mr^m.\end{aligned} \tag{2}$$

This leads to a three-term recursion relation for the coefficients given by

$$(m+2)(m+3)a_{m+2}-2(m+1)a_{m+1}+2[E+\tfrac{1}{2}-(2m+3)\lambda]a_m=0. \tag{3}$$

In order that $\phi(r)$ be an nth degree polynomial in r, it is necessary that

$$E=-\tfrac{1}{2}+(2n+3)\lambda \tag{4}$$

and

$$a_{n+1} = 0. \tag{5}$$

For $n = 0$, one just recovers the exact ground-state result $E_0 = -\frac{1}{2} + 3\lambda$, $\psi_0 = \exp(-r - \lambda r^2)$. Moreover, the structure of the recursion relation (3) is such that a_1 is always zero, so that no new result is obtained for $n = 1$ or 2. For $n \geq 3$ however, conditions (4) and (5) are together equivalent to

$$\begin{vmatrix} -4 & 12 & & & & \\ 4(n-2)\lambda & -6 & 20 & & 0 & \\ & 4(n-3)\lambda & -8 & 30 & & \\ & & & \ddots & \ddots & \\ & 0 & & & \ddots & n(n+1) \\ & & & & 4\lambda & -2n \end{vmatrix} = 0. \tag{6}$$

The structure of this determinant is such that for $n = 2l + 1$ or $2l + 2$, it leads to a polynomial equation in λ of degree l with only real and positive roots and therefore gives rise to l exact solutions. The excited state to which a particular solution $\psi(r)$ belongs depends upon the number of nodes it possesses in the region $0 \leq r \leq \infty$. In table 1 we list the energy, the coefficients a_n of the wavefunction, the value of λ and the number of nodes of the exact solutions for $3 \leq n \leq 6$. Note that one always has $a_0 = 1$, $a_1 = 0$ and that for a given n, all $a_m = 0$ for $m > n$. Also, the wavefunctions listed are arbitrary up to an overall normalisation.

This process can be continued for larger n but the algebra becomes progressively more tedious.

Case II. $\lambda < 0$

We now use

$$\begin{aligned} \psi(r) &= \phi(r) \exp(r + \lambda r^2) \\ \phi(r) &= \sum_{m=0}^{\infty} a_m r^m \end{aligned} \tag{7}$$

which leads to the three-term recursion relation

$$(m+2)(m+3)a_{m+2} + 2(m+3)a_{m+1} + 2[E + \tfrac{1}{2} + (2m+3)\lambda]a_m = 0. \tag{8}$$

The condition that $\phi(r)$ be an nth degree polynomial in r now leads to the requirement that

$$E = -\tfrac{1}{2} - (2n+3)\lambda \tag{9}$$

and

$$\begin{vmatrix} 4 & 2 & & & & \\ -4n\lambda & 6 & 6 & & 0 & \\ & -4(n-1)\lambda & 8 & 12 & & \\ & & & \ddots & \ddots & \\ & 0 & & & \ddots & n(n+1) \\ & & & & -4\lambda & 2(n+2) \end{vmatrix} = 0. \tag{10}$$

Table 1. Exact solutions for $\lambda > 0$ and $3 \le n \le 6$. The a_n are coefficients of r^n in the function $\phi(r)$, with $a_0 = 1$ and $a_1 = 0$.

	$n=3$	$n=4$	$n=5$		$n=6$	
$E+\frac{1}{2}$	9λ	11λ	13λ		15λ	
a_2	-2λ	$-\frac{8}{3}\lambda$	$-10\lambda/3$		-4λ	
a_3	$-\frac{2}{3}\lambda$	$-\frac{8}{9}\lambda$	$-10\lambda/9$		$-\frac{4}{3}\lambda$	
a_4	0	$4\lambda(4\lambda-1)/15$	$\lambda(6\lambda-1)/3$		$\frac{2}{5}\lambda(8\lambda-1)$	
a_5	0	0	$4\lambda(28\lambda-3)/135$		$8\lambda(13\lambda-1)/75$	
a_6	0	0	0		$-8\lambda(24\lambda^2-16\lambda+1)/315$	
λ	$\frac{1}{2}$	$\frac{3}{17}$	$(65-\sqrt{2929})/108$	$(65+\sqrt{2929})/108$	$(87-\sqrt{3349})/422$	$(87+\sqrt{3349})/422$
Nodes $\psi(r)$	1	1	1	2	1	2
Nodes $\tilde{\psi}(r)$	2	3	4	3	5	4

Table 2. Exact solutions for $\lambda < 0$ and $1 \le n \le 4$. $a_0 = 1$ and $a_1 = -2$.

	$n=1$	$n=2$	$n=3$		$n=4$	
$E+\frac{1}{2}$	-5λ	-7λ	-9λ		-11λ	
a_2	0	$\frac{2}{3}(2\lambda+3)$	$2(\lambda+1)$		$\frac{2}{3}(4\lambda+3)$	
a_3	0	0	$-\frac{4}{3}(2\lambda+1)$		$-\frac{2}{9}(17\lambda+6)$	
a_4	0	0	0		$(48\lambda^2+121\lambda+30)/45$	
λ	-3	$-\frac{6}{7}$	$-(13+\sqrt{109})/6$	$-(13-\sqrt{109})/6$	$-(423+\sqrt{65889})/628$	$-(423-\sqrt{65889})/628$
Nodes $\psi(r)$	1	2	2	3	3	4
Nodes $\tilde{\psi}(r)$	0	0	1	0	1	0

For $n=2l-1$ or $2l$, this determinant possesses l real and negative roots. The solutions for $1 \leqslant n \leqslant 4$ are listed in table 2. Note that one always has $a_0 = 1$ and $a_1 = -2$.

We conclude with two remarks. First, the substitution $r \to -r$ in the Hamiltonian (1) leads to

$$\tilde{H} = -\frac{1}{2}\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{1}{r} - 2\lambda r + 2\lambda^2 r^2 \tag{11}$$

which corresponds to a repulsive Coulomb potential with a polynomial perturbation. All eigenvalues of H are therefore also eigenvalues of $\tilde{H}$ provided the corresponding eigenfunctions $\tilde{\psi}(r) = \psi(-r)$ remain square integrable. Thus $\tilde{H}$ also possesses the exact ground-state solution $\tilde{E}_0 = -\frac{1}{2} + 3\lambda$ with $\tilde{\psi}_0(r) = \exp(r - \lambda r^2)$ for $\lambda > 0$. Further each of the solutions listed in the tables is also a solution of $\tilde{H}$ after the replacement $r \to -r$. The particular excited states they correspond to depends upon the number of nodes of $\tilde{\psi}(r) = \psi(-r)$ in the region $0 \leqslant r \leqslant \infty$. These are listed in the last rows of the tables.

The second remark we wish to make is that apart from the inherent interest one has in the existence of exact solutions, the results reported here are likely to be useful in perturbation calculations for the excited-state energies, particularly if the method of Dalgarno and Lewis (1955) for evaluating second-order corrections can be extended to cover such cases.

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